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CHARACTERISTIC PROPERTIES AND EXACT SOLUTIONS OF THE KINETIC EQUATION OF BUBBLY LIQUID

A. A. Chesnokov

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The paper considers a kinetic model for the motion of incompressible bubbles in an ideal liquid that takes into account their collective interaction in the case of one spatial variable. Generalized characteristics and a characteristic form of the equations are found. Necessary and sufficient hyperbolicity conditions of the integrodifferential model of rarefied bubbly flow are formulated. Exact solutions of the kinetic equation for the class of traveling waves are derived. A solution of the linearized equation is obtained.

Key words: *bubbly liquid, kinetic equations, hyperbolicity.*

Introduction. Recent studies [1–5] have developed a kinetic approach to studying the potential motion of an ideal liquid with gas bubbles based on a statistical description of the interaction of a large number of bubbles. The equations obtained taking into account the collective interaction of bubbles are similar in structure to the Vlasov equation, well known in plasma physics. The kinetic models are derived using the system of Hamilton's ordinary differential equations describing the motion of individual bubbles. To obtain this system of equations, one need to know the kinetic energy of the liquid written as the quadratic form of the bubble velocity:

$$T = 2^{-1} \sum_{i,j} \boldsymbol{u}_i^{\mathrm{t}} A_{ij} \boldsymbol{u}_j$$

($\boldsymbol{u}_i$ the velocity of the ith bubble) [6], whose coefficients are determined by the liquid flow potential in the region between the bubbles. Assuming that all bubbles are spherically imponderable particles of constant radius and using an asymptotic expansion of the solution of the Laplace equation in a small parameter (the ratio of the bubble radius to the mean interbubble distance), Russo and Smereka [3] approximately calculated the kinetic energy T and the Hamiltonian $H = 2^{-1} \sum\limits_{i,j} \boldsymbol{p}_i^{\mathrm{t}} B_{ij} \boldsymbol{p}_j$ ($\boldsymbol{p}_i = \partial T / \partial \boldsymbol{u}_i$ is the generalized momentum of the ith bubble) of bubble motion. Using the system of Hamilton's equations for the coordinates and the momenta of bubbles and the method of deriving the Vlasov equation, they obtained a kinetic equation that describes the evolution of the one-particle distribution function. Studies of [7, 8] focus on studying the characteristic properties, finding symmetries and conservation laws, and deriving exact solutions of the one-dimensional Russo–Smereka model. A kinetic model of motion for compressible bubbles in a liquid was proposed by Teshukov [5].

A drawback of the Russo–Smereka model is that an the approximately calculated Hamiltonian H of the system of equations is not a function of fixed sign. The paper of Herrero et al. [4] has a number of distinctions from [3]. In [4], in particular, the bubble velocities are more accurately expressed in terms of generalized momenta [4]: $\boldsymbol{u}_k = \sum\limits_j B_{kj} \boldsymbol{p}_j$. The matrices B_{ij} of 3×3 size are derived from the equation

$$\sum_k A_{ik} B_{kj} = I \delta_{ij}, \qquad A_{ij} = \begin{cases} (\tau \rho_l / 2) I, & i = j, \\ (3\tau \rho_l / 4)(a / |r_{ij}|)^3 (I - 3 \boldsymbol{r}_{ij} \otimes \boldsymbol{r}_{ij} / |r_{ij}|^2), & i \neq j. \end{cases}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090.
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Here I is the unit matrix, δ_{ij} is the Kronecker symbol, τ and a are the bubble volume and radius, respectively, ρ_l is the density of the liquid, $\boldsymbol{a}\otimes\boldsymbol{b}$ is the notation of the dyad, and $\boldsymbol{r}_{ij}=\boldsymbol{x}_i-\boldsymbol{x}_j$ ($\boldsymbol{x}_i$ are the coordinates of the center of the ith bubble). In this case, the Hamiltonian of the system is a nonnegative function. As a result, the following kinetic model of rarefied bubbly flow was proposed [4]:

$$f_t+\operatorname{div}_{\boldsymbol{x}}(H_{\boldsymbol{p}}f)-\operatorname{div}_{\boldsymbol{p}}(H_{\boldsymbol{x}}f)=0,\qquad t\geqslant 0,\quad \boldsymbol{x}\in R^3,\quad \boldsymbol{p}\in R^3,$$

$$H(t,\boldsymbol{x},\boldsymbol{p})=2^{-1}|\boldsymbol{p}+\nabla\varphi|^2,\qquad \operatorname{div}_{\boldsymbol{x}}((1+\lambda\rho)\nabla\varphi(t,\boldsymbol{x}))+\lambda\operatorname{div}_{\boldsymbol{x}}\boldsymbol{j}(t,\boldsymbol{x})=0, \tag{1}$$

$$\rho(t,\boldsymbol{x})=\int\limits_{R^3}f(t,\boldsymbol{x},\boldsymbol{p})\,d\boldsymbol{p},\qquad \boldsymbol{j}(t,\boldsymbol{x})=\int\limits_{R^3}\boldsymbol{p}f(t,\boldsymbol{x},\boldsymbol{p})\,d\boldsymbol{p}.$$

Here $f(t,\boldsymbol{x},\boldsymbol{p})$ is the density of bubble distribution over the Cartesian coordinates $\boldsymbol{x}$ and the bubble momenta $\boldsymbol{p}$ at time t and λ is a constant (all values are considered dimensionless). In the steady case, the curves $H=\text{const}$ are integral curves of system (1). By virtue of the inequality $H(t,\boldsymbol{x},\boldsymbol{p})>0$, the integral curves in the phase space $(\boldsymbol{x},\boldsymbol{p})$ are closed and bounded. Below, we consider Eq. (1) in the one-dimensional case.

1. Generalized Characteristics and Hyperbolicity Conditions of the Kinetic Equation. In the case of one spatial variable, Eq. (1) becomes

$$f_t+(p-l)f_x+(p-l)l_xf_p=0, \tag{1.1}$$

where

$$l=\frac{\lambda j}{1+\lambda\rho},\qquad \rho=\int\limits_{-\infty}^{\infty}f\,dp,\qquad j=\int\limits_{-\infty}^{\infty}pf\,dp.$$

The distribution density f is considered a nonnegative function that is finite or rapidly decreasing as $|p|\to\infty$. Equation (1.1) is close in structure to the one-dimensional Russo–Smereka equation

$$f_t+(p-\lambda j)f_x+\lambda pj_xf_p=0,\qquad j=\int\limits_{-\infty}^{\infty}pf\,dp$$

and is converted into this equation if we use the expansion of the function $l(t,x;\lambda)$ in powers of λ, ignoring terms of the order of λ^2 and higher. Below, without loss of generality, we set $\lambda=1$ (this is achieved by extension of the variable f). In Eq. (1.1), we convert to the Eulerian–Lagrangian coordinates x and ν by substituting the variable $p=p(t,x,\nu)$, where the function p is a solution of the Cauchy problem

$$p_t+(p-l)p_x=(p-l)l_x,\qquad p\Big|_{t=0}=p_0(x,\nu).$$

In the variables t, x, and ν from Eq. (1.1), we obtain the following system for the functions p and f:

$$p_t+(p-l)(p-l)_x=0,\qquad f_t+(p-l)f_x=0. \tag{1.2}$$

Here

$$l=\frac{j}{1+\rho},\qquad \rho=\int\limits_{-\infty}^{\infty}p_\nu f\,d\nu,\qquad j=\int\limits_{-\infty}^{\infty}pp_\nu f\,d\nu.$$

The substitution is invertible if $p_\nu(t,x,\nu)\neq 0$. In this case, we can determine $\nu(t,x,p)$ and, after substitution into $f(t,x,\nu)$, obtain a solution $f=f(t,x,p)$ of Eq. (1.1). Below, we assume that the inequality $p_\nu>0$ holds.

Equations (1.2) belong to the class of systems with the operator coefficients

$$\boldsymbol{U}_t+A\boldsymbol{U}_x=0,\qquad \boldsymbol{U}=\boldsymbol{U}(t,x,\nu), \tag{1.3}$$

for which the notions of characteristics and hyperbolicity are generalized [9]. In this case, $\boldsymbol{U}=(p,f)^{\mathrm{t}}$ and A is a nonlocal matrix operator acting over the variable ν:

$$A=\begin{pmatrix}(p-l)+(1+\rho)^{-1}\displaystyle\int\limits_{-\infty}^{\infty}(p-l)f_\nu\ldots\,d\nu & -(1+\rho)^{-1}\displaystyle\int\limits_{-\infty}^{\infty}(p-l)p_\nu\ldots\,d\nu\\ 0 & p-l\end{pmatrix}.$$

Let B be a Banach space of the vector-functions $\boldsymbol{\varphi} = (\varphi_1(\nu), \varphi_2(\nu))^{\mathrm{t}}$ and the functions $\boldsymbol{U}$, $\boldsymbol{U}_t$, and $\boldsymbol{U}_x$ (dependence on t and x as parameters) belong to B. It can easily be shown that A: $B \to B$ is a linear operator. We consider the eigenvalue problem

$$(\boldsymbol{F}, (A - kI)\boldsymbol{\varphi}) = 0. \tag{1.4}$$

The solution is sought for the vector functional $\boldsymbol{F} \in B^*$ (B^* is a space of functionals that is conjugate with B) acting on an arbitrary vector-function $\boldsymbol{\varphi} \in B$. The expression $(\boldsymbol{F}, \boldsymbol{\varphi})$ denotes the result from the action of the functional on a test vector-function. The action of the functional $\boldsymbol{F}$ on Eq. (1.3) yields the relation on the characteristic

$$(\boldsymbol{F}, \boldsymbol{U}_t + k\boldsymbol{U}_x) = 0. \tag{1.5}$$

According to [9], system (1.3) is hyperbolic if all eigenvalues of k are real and the corresponding eigenfunctionals form a complete system, i.e., the set of relations on characteristics (1.5) is equivalent to Eqs. (1.3).

For system (1.2), the eigenvalue and functional problem (1.4) yields the following two equations for determining $\boldsymbol{F} = (F_1, F_2)$:

$$\begin{gathered}(F_1, (p - l - k)\varphi_1) + (1 + \rho)^{-1} \int\limits_{-\infty}^{\infty} (p - l) f_\nu \varphi_1 \, d\nu (F_1, (p - l)) = 0, \\ (F_2, (p - l - k)\varphi_2) - (1 + \rho)^{-1} \int\limits_{-\infty}^{\infty} (p - l) p_\nu \varphi_2 \, d\nu (F_1, (p - l)) = 0.\end{gathered} \tag{1.6}$$

In this case, we used the independence of the components on the trial function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^{\mathrm{t}}$. We note that problem (1.6) has a continuous spectrum of characteristic velocities $k^\nu = p(\nu) - l$, which correspond to the eigenfunctionals $\boldsymbol{F}^{1\nu}$ and $\boldsymbol{F}^{2\nu}$ with the components

$$(F_1^{1\nu}, \varphi) = \frac{\varphi}{v} - \int\limits_{-\infty}^{\infty} \frac{v' f'_\mu (\varphi' - \varphi) \, d\mu}{v' - v}, \qquad (F_2^{1\nu}, \varphi) = \int\limits_{-\infty}^{\infty} \frac{v' v'_\mu (\varphi' - \varphi) \, d\mu}{v' - v},$$

$$(F_1^{2\nu}, \varphi(\mu)) = 0, \qquad (F_2^{2\nu}, \varphi(\mu)) = \varphi(\nu)$$

defined on smooth trial functions. Here $v = p(\nu) - l$ and $v' = v(\mu)$ (arguments t and x are omitted for brevity).

To define the discrete characteristic spectrum, we consider the case $k(t, x) \neq p(t, x, \nu) - l(t, x)$ in a region in which $f(t, x, \nu) > 0$. System (1.6) has the solution

$$(F_1, \varphi) = -\int\limits_{-\infty}^{\infty} \frac{(p - l) f_\nu \varphi \, du}{p - l - k}, \qquad (F_2, \varphi) = \int\limits_{-\infty}^{\infty} \frac{(p - l) p_\nu \varphi \, du}{p - l - k},$$

where $k(t, x)$ is a root of the equation

$$\chi(k) = 1 + k^2 \int\limits_{-\infty}^{\infty} \frac{p_\nu f \, d\nu}{(p - l - k)^2} = 0. \tag{1.7}$$

Equation (1.7), defining the discrete characteristic spectrum of the operator A, has no real roots. Indeed, if f is not a finite function, k does not belong to the real axis by virtue of the condition $k \neq p - l$. For the case of a finite distribution function, this follows from the type of the function $\chi(k)$ and the inequalities $f \geqslant 0$ and $p_\nu \geqslant 0$. However, Eq. (1.7) can have complex roots, and for hyperbolicity of the system, one need to formulate conditions of the absence of such roots.

Acting on Eqs. (1.2) by the eigenfunctionals $\boldsymbol{F}^{1\nu}$ and $\boldsymbol{F}^{2\nu}$, we obtain the characteristic form of the system:

$$\begin{gathered}v^{-2}(\partial_t + v\partial_x)p + v^{-1}(\partial_t + v\partial_x)\rho + (\partial_t + v\partial_x) \int\limits_{-\infty}^{\infty} \frac{p'_\mu f' \, d\mu}{p' - p} = 0, \\ f_t + v f_x = 0.\end{gathered} \tag{1.8}$$

Obviously, model (1.2) does not reduce to the Riemann invariants, which are preserved along the characteristics of the continuous spectrum, unlike the Russo–Smereka equations [7].

The hyperbolicity conditions for system (1.2) are formulated in terms of the complex function $\chi(z)$, or, more precisely, its limiting values from the upper and lower half planes on the real axis

$$\chi^{\pm}(v(\nu)) = 1 + v^2 \int_{-\infty}^{\infty} \frac{f'_\mu \, d\mu}{v' - v} \pm \pi i v^2 \frac{f_\nu}{v_\nu}.$$

We note that the function $\chi(z)$ is analytical and has no poles in the complex plane with the real axis cut out. The argument principle leads to the following lemma.

Lemma 1. *On the solution* $p(t, x, \nu)$, $f(t, x, \nu)$, *Eq.* (1.7) *has no complex roots if the following conditions are satisfied:*

$$\Delta \arg \chi^{\pm} = 0, \qquad \chi^{\pm} \neq 0 \tag{1.9}$$

(*the increment of the argument is calculated for* ν *varying from* $-\infty$ *to* ∞ *and fixed values of* t *and* x).

The lemma below specifies conditions under which the relations on characteristics (1.8) are equivalent to Eqs. (1.2).

Lemma 2. *Let the components of the vector-function* $\boldsymbol{S} = (S_1, S_2)^{\rm t}$ *be of the Hölder type in the variable* ν *and the relations* $(\boldsymbol{F}^{1\nu}, \boldsymbol{S}) = 0$ *and* $(\boldsymbol{F}^{2\nu}, \boldsymbol{S}) = 0$ *and conditions* (1.9) *hold. Then, we have* $\boldsymbol{S} \equiv 0$.

Proof. It follows from the equation $(\boldsymbol{F}^{2\nu}, \boldsymbol{S}) = 0$ that $S_2 \equiv 0$. In view of this fact, the equation $(\boldsymbol{F}^{1\nu}, \boldsymbol{S}) = 0$ can be written as

$$S_1 - v \int_{-\infty}^{\infty} \frac{v' f'_\mu (S'_1 - S_1) \, d\mu}{v' - v} = 0.$$

Converting to the integration variable v', we obtain the singular integral equation

$$S_1 \Big(1 + v \int_{-\infty}^{\infty} \frac{v' f'_{v'} \, dv'}{v' - v} \Big) - v \int_{-\infty}^{\infty} \frac{v' f'_{v'} S'_1 \, dv'}{v' - v} = 0. \tag{1.10}$$

We reduce Eq. (1.10) to the conjugation problem of the theory of analytical functions [10]. We introduce the function

$$\Psi(z) = z \int_{-\infty}^{\infty} \frac{v' f'_{v'} S'_1 \, dv'}{v' - z}.$$

Using the Sokhotskii–Plemelj formulas, one can calculate the limiting values of the function $\Psi(z)$ on the real axis. It can be easily verified that the solution of Eq. (1.10) is equivalent to the solution of the homogeneous conjugation problem

$$\Psi^+(v) = G(v) \Psi^-(v), \qquad G(v) = \chi^+(v)/\chi^-(v), \tag{1.11}$$

where G is a specified function and Ψ is the desired function. It is required to determine the analytical functions of the complex variable in the upper and lower half-planes from the boundary condition on the real axis. By virtue of satisfaction of conditions (1.9), the index of the conjugation problem is zero. In this case, according to [10], problem (1.11) has only a trivial solution in the class of functions vanishing at infinity. From Eq. (1.10), taking into account that $\Psi = \Psi^{\pm} = 0$, we obtain the equalities $\operatorname{Re}(\chi^{\pm} S_1) = 0$ and $\operatorname{Im}(\chi^{\pm} S_1) = 0$. Since $\chi^{\pm} \neq 0$, we require that $S_1 = 0$. Thus, the completeness of the system of eigenfunctionals is proved. Lemma 2 is proved.

The above lemmas and the definition of hyperbolicity leads to the following theorem.

Theorem 1. *Conditions* (1.9) *are necessary and sufficient for hyperbolicity of system* (1.2) *if the functions* p *and* f *are differentiable and* p_ν *and* f_ν *are Hölder functions in the variable* ν ($p_\nu \neq 0$), *and* f *and* f_ν *vanish as* $|\nu| \to \infty$.

2. Traveling Waves. A solution of the form $f = f(\xi, p)$, where $\xi = x - Dt$, describes a traveling wave propagating at constant velocity D. For this class of solutions, Eq. (1.1) takes the form

$$(p - l - D) f_\xi + (p - l) l_\xi f_p = 0.$$

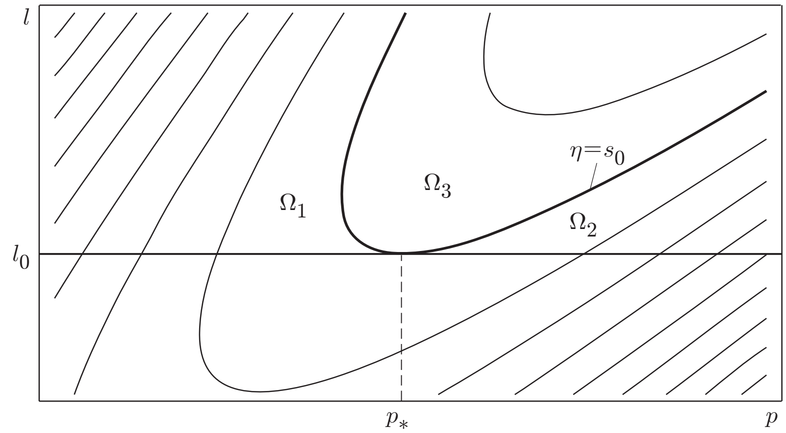

Fig. 1

As an independent variable, it is convenient to use l (we set $l_\xi \neq 0$). Then, the traveling wave equation is written as

$$(p - l - D)f_l + (p - l)f_p = 0. \tag{2.1}$$

Equation (2.1) is integrated:

$$f = \Phi(\eta), \qquad \eta = p^2/2 - (D + l)p + l^2/2. \tag{2.2}$$

Let $D > 0$ (the case of $D < 0$ is similar, and for $D = 0$, the solution is easily constructed by the characteristics method). Solution (2.2) takes constant values on the characteristics $\eta = \text{const}$ [these lines in the plane of the variables (p, l) are shown in Fig. 1].

Let us consider the Cauchy problem

$$f(l_0, p) = f_0(p), \qquad l_0 = \Big(1 + \int\limits_{-\infty}^{\infty} f_0(p)\,dp\Big)^{-1} \int\limits_{-\infty}^{\infty} p f_0(p)\,dp. \tag{2.3}$$

Conditions (2.3) ensures that the traveling wave is continuously contiguous to the specified steady-state, spatially homogeneous solution $f_0(p)$. As can be seen from Fig. 1, for $l > l_0$, the solution of the Cauchy problem is uniquely determined from the initial data in the domains Ω_1 and Ω_2, whereas in the domain Ω_3 bounded by the curve $\eta = s_0 = -D^2/2 - Dl_0$, the solution is found from additional equations. We note that the Cauchy problem (2.3) is incorrect for $l < l_0$ because the function $f_0(p)$ cannot be specified arbitrarily (the characteristics intersect the line on which the Cauchy conditions are specified at two points). Let us find the solution of problem (2.1) and (2.3) in a certain interval $l_0 \leqslant l \leqslant l_1$.

We determine the function $\Phi(\eta)$ on the line $l = l_0$ for $p \geqslant p_* = l_0 + D$ and $p \leqslant p_*$. We denote the values of the function Φ on these intervals by Φ^+ and Φ^-, respectively. Formulas (2.2) and (2.3) yield

$$\Phi^+(\eta) = f_0(l_0 + D + \sqrt{2\eta + D^2 + 2Dl_0}), \qquad \Phi^-(\eta) = f_0(l_0 + D - \sqrt{2\eta + D^2 + 2Dl_0}).$$

Using the known functions $\Phi^\pm$, from (2.2) we obtain the following solution in the domains Ω_1 and Ω_2:

$$\begin{aligned} f &= \Phi^+(p^2/2 - (l + D)p + l^2/2), \qquad p \geqslant l + D + \sqrt{2D(l - l_0)}, \\ f &= \Phi^-(p^2/2 - (l + D)p + l^2/2), \qquad p \leqslant l + D - \sqrt{2D(l - l_0)}. \end{aligned} \tag{2.4}$$

To construct the solution in the domain Ω_3, we transform the relation

$$l = \Big(1 + \int\limits_{-\infty}^{\infty} f\,dp\Big)^{-1} \int\limits_{-\infty}^{\infty} pf\,dp \tag{2.5}$$

to an integral equation for the function $\Phi(\eta)$ in the interval $\eta \in (s, s_0)$, where $s = -D^2/2 - Dl$. For this, we convert

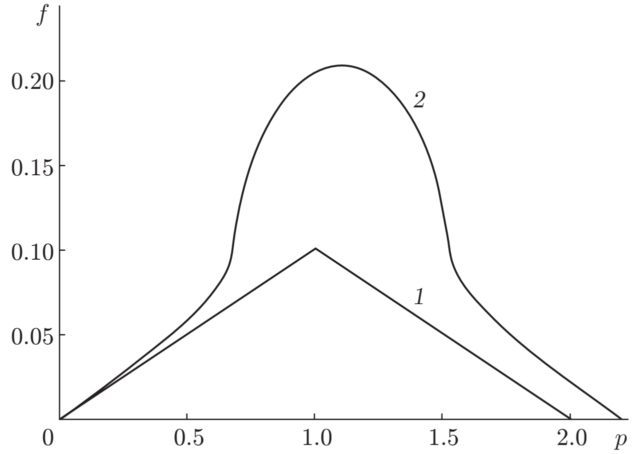

Fig. 2

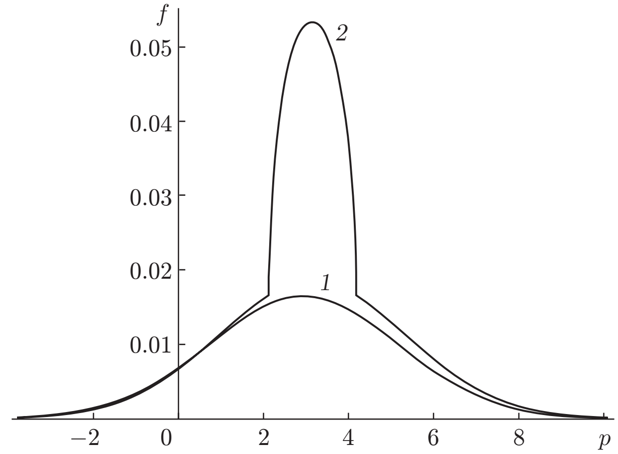

Fig. 3

from the integration variable p in Eq. (2.5) to the variable η. As a result, we obtain the integral Abel equation for the function $\Phi(\eta)$ in the domain Ω_3:

$$\int_s^{s_0} \frac{\Phi(\eta)\,d\eta}{\sqrt{\eta - s}} = F(s). \tag{2.6}$$

Here

$$F(s) = -\frac{1}{2\sqrt{2}} - \frac{s}{D^2\sqrt{2}} - \frac{1}{2}\int_{s_0}^{\infty} \frac{\Phi^+(\eta) + \Phi^-(\eta)}{\sqrt{\eta - s}}\,d\eta - \frac{1}{D\sqrt{2}}\int_{s_0}^{\infty} (\Phi^+(\eta) - \Phi^-(\eta))\,d\eta.$$

Solution of the Abel equation (2.6) yields

$$\Phi(\eta) = \frac{1}{\pi}\Big(\frac{F(s_0)}{\sqrt{s_0 - \eta}} - \int_{\eta}^{s_0} \frac{F'(s)\,ds}{\sqrt{s - \eta}}\Big).$$

In the domain Ω_3, the solution is written as

$$f = \Phi(p^2/2 - (l + D)p + l^2/2), \tag{2.7}$$

where $l + D - \sqrt{2D(l - l_0)} \leqslant p \leqslant l + D + \sqrt{2D(l - l_0)}$. Thus, formulas (2.4) and (2.7) completely define the solution in the traveling-wave region.

Let us give two examples of constructing a traveling wave using the algorithm described above. In the first example (Fig. 2), the initial distribution is a finite function of the form

$$f_0(p) = \begin{cases} hp, & 0 \leqslant p \leqslant 1, \\ h(2 - p), & 1 \leqslant p \leqslant 2, \\ 0, & \text{otherwise.} \end{cases}$$

In the second example (Fig. 3), we have

$$f_0(p) = \rho_0(2\pi T)^{-1/2} \exp\,(-(p - a)^2/(2T)).$$

In this case, the solution can be obtained in explicit form

$$f = \rho_0(2\pi T)^{-1/2} \exp(-(p^2 - 2(l + D)p + l^2 + D^2 + 2Dl_0)/(2T)), \qquad p \notin (p_1, p_2),$$
$$f = \Phi(p^2/2 - (l + D)p + l^2/2), \qquad p_1 \leqslant p \leqslant p_2,$$

$$\Phi(\eta) = \frac{\sqrt{2}}{\pi D^2}\sqrt{s_0 - \eta} - \frac{\rho_0}{\sqrt{2\pi T}} \exp\Big(-\frac{D^2 + 2Dl_0 + 2\eta}{2T}\Big)\Big(\operatorname{erf}\Big(\sqrt{\frac{s_0 - \eta}{T}}\Big) - 1\Big),$$

$$p_1 = l + D - \sqrt{2D(l - l_0)}, \qquad p_2 = l + D + \sqrt{2D(l - l_0)}.$$

In Figs. 2 and 3, curves 1 show a plot of the initial distribution function; curves 2 correspond to the solution for fixed $l > l_0$.

From the traveling wave equation (2.1), it follows that $\rho = \rho_0 + d^{-1}(l - l_0)$ and that the bubble concentration increases in the wave region. This explains an abrupt increase in the function f (see Figs. 2 and 3). In this case, rotational motion of the bubbles occurs in the traveling-wave region. Indeed, the trajectories of bubble motion are defined as solutions of the system $dl/dt = p - l - D$ and $dp/dt = p - l$. In the domain Ω_3, the quantity $p - l - D$ changes sign, and at some points, the bubble velocity coincides with the wave velocity.

3. Solution of the Linearized Kinetic Equation. We linearize Eqs. (1.2) on the steady-state, spatially homogeneous solution $p = p^0(\nu)$, $f = f^0(\nu)$. As a result, for small perturbations, we have the system

$$p_t + (p^0 - l^0)(p - l)_x = 0, \qquad f_t + (p^0 - l^0)f_x = 0. \tag{3.1}$$

Here

$$l = \frac{1}{1+\rho^0} \int\limits_{-\infty}^{\infty} (pp_\nu^0 f^0 + p^0 p_\nu f^0 + p^0 p_\nu^0 f)\, d\nu - \frac{l^0}{1+\rho^0} \int\limits_{-\infty}^{\infty} (p_\nu^0 f + p_\nu f^0)\, d\nu,$$

$$l^0 = \frac{j^0}{1+\rho^0}, \qquad \rho^0 = \int\limits_{-\infty}^{\infty} p_\nu^0 f^0\, d\nu, \qquad j^0 = \int\limits_{-\infty}^{\infty} p^0 p_\nu^0 f^0\, d\nu.$$

The characteristic form of system (3.1) is obtained by linearizing relations (1.8). Using the notation

$$W(t,x,\nu) = \frac{p(t,x,\nu)}{(v^0(\nu))^2} + \int\limits_{-\infty}^{\infty} \frac{p_\mu(t,x,\mu) f^0(\mu) - p_\mu^0(\mu) f(t,x,\mu)}{p^0(\mu) - p^0(\nu)}\, d\mu$$

$$+ \frac{1}{v^0(\nu)} \int\limits_{-\infty}^{\infty} (p_\nu^0 f + p_\nu f^0)\, d\nu - \int\limits_{-\infty}^{\infty} \frac{(p(t,x,\mu) - p(t,x,\nu)) p_\mu^0(\mu) f^0(\mu)}{(p^0(\mu) - p^0(\nu))^2}\, d\mu \quad (v^0(\nu) = p^0(\nu) - l^0) \tag{3.2}$$

the relations on the characteristics for Eqs. (3.1) take the form

$$W_t + v^0(\nu) W_x = 0, \qquad f_t + v^0(\nu) f_x = 0. \tag{3.3}$$

We consider the Cauchy problem $p\Big|_{t=0} = p_0(x,\nu)$ and $f\Big|_{t=0} = f_0(x,\nu)$. If the functions p_0 and f_0 are specified, the function $W_0(x,\nu)$ is known by virtue of (3.2). Integration of Eqs. (3.3) with the initial conditions yields the solution

$$W = W_0(x - v^0(\nu)t, \nu), \qquad f = f_0(x - v^0(\nu)t, \nu).$$

To derive the solution of the initial equations (3.1), it is necessary to find the function $p(t,x,\nu)$ from the known functions f and W. This is possible by solving a singular integral equation. Several simple transformations of Eq. (3.2) yield the equation for $p(t,x,\nu)$ (dependence on t and x as parameters)

$$\Big(\frac{1}{v^0} + v^0 \int\limits_{-\infty}^{\infty} \frac{f_\mu^0(\mu)\, d\mu}{v^0(\mu) - v^0(\nu)}\Big) p - \int\limits_{-\infty}^{\infty} \frac{v^0(\mu) f_\mu^0(\mu) p'}{v^0(\mu) - v^0(\nu)} = v^0 W - \int\limits_{-\infty}^{\infty} v_\mu^0 f'\, d\mu + \int\limits_{-\infty}^{\infty} \frac{v_\mu^0(\mu) f'\, d\mu}{v^0(\mu) - v^0(\nu)}. \tag{3.4}$$

Let $z = v^0(\nu)$ on the real axis. We denote $\xi = v^0(\mu)$ and $\bar{f}^0(\xi) = f^0(\mu)$ and introduce the complex functions

$$\chi(z) = 1 + z^2 \int\limits_{-\infty}^{\infty} \frac{\bar{f}_\xi^0\, d\xi}{\xi - z}, \qquad \Psi(z) = \frac{1}{2} \int\limits_{-\infty}^{\infty} \frac{\xi \bar{f}_\xi^0 p\, d\xi}{\xi - z}.$$

The function $\chi(z)$ coincides with the characteristic function defined in Sec. 2. The hyperbolicity conditions (1.9) are formulated in terms of the limiting values of this function. Let conditions (1.9) be satisfied on the solution. Using the Sokhotskii–Plemelj formulas, it is easy to show that the solution of the singular integral equation (3.4) reduces to the solution of the inhomogeneous conjugate problem

$$\Psi^+(z) = G(z)\Psi^-(z) + g(z), \qquad G(z) = \chi^+(z)/\chi^-(z), \tag{3.5}$$

where

$$g(z) = \frac{\pi i z^2 \bar{f}_z^0}{\chi^-(z)} \Big(zW - \int\limits_{-\infty}^{\infty} \bar{f}\, d\xi + \int\limits_{-\infty}^{\infty} \frac{\bar{f}\, d\xi}{\xi - z} \Big).$$

If the hyperbolicity conditions (1.9) are satisfied, the index of the conjugation problem is equal to zero and in the class of functions disappearing at infinity, solution (3.5) has the following form [10]:

$$\Psi(z) = \frac{\chi(z)}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{g(\xi)\, d\xi}{\chi^+(\xi)(\xi - z)}.$$

The function p is found by the formula $p(z) = (\Psi^+(z) - \Psi^-(z))/(\pi i z f_z^0)$. After calculations and an inverse substitution of variables, we obtain

$$p(t,x,\nu) = v^0(\nu)\Big(m(t,x,\nu)\chi(v^0(\nu)) + \int\limits_{-\infty}^{\infty} \frac{(v^0(\mu))^2 f_\mu^0(\mu) m(t,x,\mu)\, d\mu}{v^0(\mu) - v^0(\nu)}\Big),$$

$$m(t,x,\nu) = \frac{1}{\chi^+\chi^-}\Big(v^0(\nu)W(t,x,\nu) - \int\limits_{-\infty}^{\infty} f p_\mu^0(\mu)\, d\mu + \int\limits_{-\infty}^{\infty} \frac{v_\mu^0(\mu) f(t,x,\mu)\, d\mu}{v^0(\mu) - v^0(\nu)}\Big).$$

Thus, the functions f and p are determined and the solution of the linearized kinetic equation (3.1) is constructed.

Conclusion. The one-dimensional kinetic equation for a bubble liquid studied by Herrero et al. [4] was analyzed theoretically. It was established that this equation reduces to an integrodifferential system, which is generalized hyperbolic [9] under certain conditions. Exact solutions of the model were constructed in the class of traveling waves. A solution is obtained to the kinetic equation linearized on a steady-state, spatially homogeneous distribution function. Unlike the Russo–Smereka equation, which is similar in structure, this model has a Hamiltonian of fixed sign but obviously fewer symmetry properties: it cannot be reduced to the Riemann invariant and nor has any infinite series of conservation laws.

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